Recall: Solving for y_k in Euler's method for $y' = f(t)$ with $y(a) = 0$ converts "$\int$" to be matrix

Given: $y_0 = 0$

$$\left\{\begin{aligned} y_1 &= \cancel{y_0}^{0} + f_0 \cdot h = f_0 \cdot h && + \cancel{y_0}^{0} \\ y_2 &= y_1 + f_1 \cdot h = (f_0 + f_1)\cdot h && + \cancel{y_0}^{0} \\ y_3 &= y_2 + f_2 \cdot h = (f_0 + f_1 + f_2)\cdot h && + \cancel{y_0}^{0} \\ &\vdots \qquad\qquad \vdots \\ y_{k+1} &= y_k + f_k \cdot h = (f_0 + \cdots + f_k)\cdot h && + \cancel{y_0}^{0} \end{aligned}\right\}$$

as matrix eqn.

already known $\rightarrow \cancel{y_0}$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n+1} \end{bmatrix}}_{y} \; = \; \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}_{\int} \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{f} \cdot \underbrace{h}_{dt}$$

Since $y = \int f\, dt$, we can write $y_k = F_k$ to get (pointwise) anti-derivative formula:

$$F_k = (f_0 + \cdots + f_{k-1}) \cdot h$$ ⟵ (left-endpoint Riemann sum)

Reverse: Solving for f_k in Euler's method for $y' = f(t)$ with $y(a) = 0$ converts "$\frac{d}{dt}$" to matrix

Given: $y_0 = 0$

$$\begin{aligned} y_1 &= \cancel{y_0} + f_0 \cdot h &\rightsquigarrow\quad & \tfrac{1}{h}(y_1 - \cancel{y_0}^{0}) = f_0 \\ y_2 &= y_1 + f_1 \cdot h &\rightarrow\quad & \tfrac{1}{h}(y_2 - y_1) = f_1 \\ y_3 &= y_2 + f_2 \cdot h &\rightarrow\quad & \tfrac{1}{h}(y_3 - y_2) = f_2 \\ &\vdots & & \qquad\vdots \\ y_{k+1} &= y_k + f_k \cdot h &\rightsquigarrow\quad & \tfrac{1}{h}(y_{k+1} - y_k) = f_k \end{aligned}$$

as matrix eqn.

$\cancel{y_0}$ ⟵ already known

$$\underbrace{\frac{1}{h}\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}}_{d/dt} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n+1} \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}}_{f}$$

Since $y' = f$, we can write $y'_k = f_k$ to get (pointwise) derivative formula:

$$y'_k = \tfrac{1}{h}(y_{k+1} - y_k)$$ ⟵ ("forward difference")

"Discretizing a differential equation" will mean converting it to a matrix equation

$$D\,y = f$$

D = matrix of numbers $\qquad y = \begin{pmatrix}\text{unknown}\\ \text{vector}\end{pmatrix}$

f = vector of numbers

EX: Discretize $y' = 2t - 1$ with $y(2) = 0$ on $[2, 3]$ with step-size $h = 1/3$.

$t_0 = 2$ $\qquad y_0 = y(t_0) = y(2) = 0$

$t_1 = 7/3$

$t_2 = 8/3$

$t_3 = 3$

$\left.\begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix}\right\}$ unknown

Use the pointwise deriv. formula to write a system of equations for y_1, y_2, y_3

$$\frac{1}{1/3}(y_1 - \cancel{y_0}^{\,0}) = y_0' = 2t_0 - 1 = 4 - 1 = 3$$

$$\frac{1}{1/3}(y_2 - y_1) = y_1' = 2t_1 - 1 = 14/3 - 1 = 11/3$$

$$\frac{1}{1/3}(y_3 - y_2) = y_2' = 2t_2 - 1 = 16/3 - 1 = 13/3$$

There is no y_4!

$$\cancel{\frac{1}{1/3}(y_4 - y_3) = y_3' = 2t_3 - 1 = 6 - 1 = 5}$$

Ans:

$$3\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11/3 \\ 13/3 \end{bmatrix}$$

Since we know that $\frac{d}{dt}$ as a matrix is

$$1/h \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

we can immediately give answers to simple problems

EX Discretize $y' = t^2$ with $y(-1) = 0$ on $[-1, 1]$ with step-size $h = 1/2$.

$t_0 = -1$	$f_0 = 1$
$t_1 = -1/2$	$f_1 = 1/4$
$t_2 = 0$	$f_2 = 0$
$t_3 = 1/2$	$f_3 = 1/4$
$t_4 = 1$	$f_4 = 1$

Ans:

$$\underbrace{\frac{1}{1/2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{d/dt}\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1 \\ 1/4 \\ 0 \\ 1/4 \end{bmatrix}}_{f}$$

Notation: $\begin{bmatrix} 1 & 0 & 0 & \\ -1 & 1 & 0 & \\ 0 & -1 & 1 & \ldots \end{bmatrix}$ is the "forward difference" matrix

$= \Delta^+$ "Delta-plus"

$y_k' = 1/h\,(y_{k+1} - y_k)$ is the (pointwise) "forward difference" deriv.

→ These calculate the derivative of y starting with a known value $\begin{pmatrix}\text{moving}\\ \text{forwards}\end{pmatrix}$

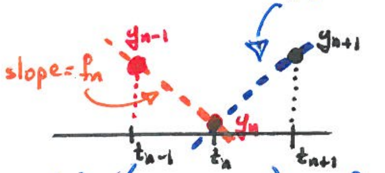
There is also a "backward difference" coming from Euler's method with known ending value.

$$y' = f(t) \quad \text{on } [a,b] \quad \text{with} \quad y(b) = 0$$

Given: $y_{n+1} = 0$ ← last y-value

$y_n = y_{n+1} + f_{n+1}\cdot(-h)$ ← move left by h

$y_{n-1} = y_n + f_n\cdot(-h)$



$\vdots$

$$y_{k-1} = y_k + f_k\cdot(-h) \rightsquigarrow \frac{1}{h}\left(y_k - y_{k-1}\right) = f_k$$

(Pointwise) Backward Difference

$$y_k' = \frac{1}{h}\left(y_k - y_{k-1}\right)$$

Backward Difference Matrix

"Delta minus" → $\Delta^- = \begin{bmatrix} -1 & 1 & 0 & \\ 0 & -1 & 1 & \\ 0 & 0 & -1 & \\ & & & \ddots \end{bmatrix}$

$$\frac{1}{h}\begin{bmatrix} -1 & 1 & 0 & \cdots & \cdots & 0 & \\ 0 & -1 & 1 & \cdots & \cdots & 0 & \\ \vdots & & \ddots & \ddots & & \vdots & \\ 0 & \cdots & 0 & -1 & 1 & & \\ 0 & \cdots & \cdots & 0 & 0 & -1 & \cancel{1} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n+1} \end{bmatrix} \quad \cancel{f_0}$$

$\cancel{y_{n+1}}$ ← already known

EX: Discretize $y' = t^2$ with $y(1) = 0$ on $[-1, 1]$ with step-size $h = 1/2$.

Given the value of y at end → must work backwards.

Ans:

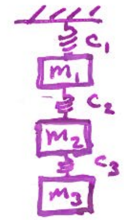
$$\underbrace{\frac{1}{1/2}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{d/dt} \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 1/4 \\ 0 \\ 1/4 \\ 1 \end{bmatrix}}_{f}$$

$\cancel{1} = f_0$

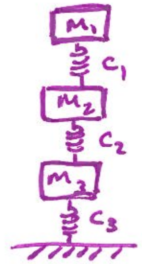
$\cancel{y_4} = y(1) = 0$

Note: Forward & Backward Difference matrices are elongation matrices for spring systems!

$\Delta^+ = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ elongation matrix of



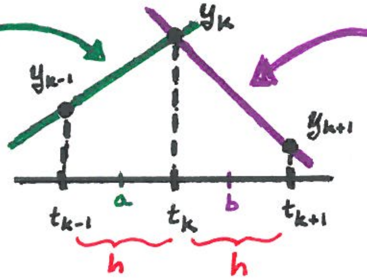
$\Delta^- = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ elongation matrix of



Geometric meaning of "forward" / "backward" diff.

"Backward" diff. says this slope is y_k'

"Incoming" slope at y_k



"Forward" diff. says this slope is y_k'

"Outgoing" slope at y_k

Note that probably neither forward nor backward difference give the correct value for $y'(t_k)$!

But the Mean Value Theorem from calculus tells us that

- Backward diff $= y'(a)$ for some a between t_{k-1} and t_k
- Forward diff $= y'(b)$ for some b between t_k and t_{k+1}

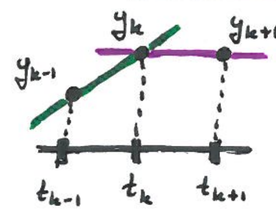
If h is small, then $a \approx t_k$ and $t_k \approx b$ — so hopefully $y'(a) \approx y'(t_k) \approx y'(b)$

... Actually size of error here is measured by $\frac{d^2}{dt^2}$. When $y''(t_k)$ is big (like in the picture above) even a small h can lead to bad approximations.

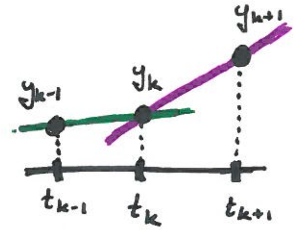
Centered differences.

In the example picture on the left, the "backward" difference is too big and the "forward" difference is too small.

→ Always one will be too big and the other too small:



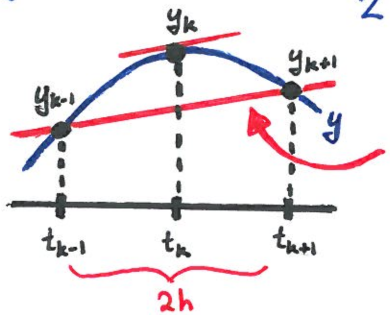
Backward too big
Forward too small



Backward too small
Forward too big

Solution: Use average of forward & backward!

$$y_k' = \frac{\frac{1}{h}(y_{k+1} - \cancel{y_k}) + \frac{1}{h}(\cancel{y_k} - y_{k-1})}{2} = \frac{1}{2h}(y_{k+1} - y_{k-1})$$



"Centered" diff. says this slope is y_k'

"Average" slope at y_k

(Pointwise) Centered Difference

$$y_k' = \frac{1}{2h}(y_{k+1} - y_{k-1})$$

Unfortunately, centered differences **cannot be used to solve 1^{st} order equations**

$$\underline{y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} \xrightarrow{\text{centered diff}} \begin{bmatrix} \cancel{y_0'} \\ y_1' \\ \vdots \\ y_n' \\ \cancel{y_{n+1}'} \end{bmatrix} = \frac{1}{2h} \begin{bmatrix} -1 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & -1 & 0 & 1 & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ & & & -1 & 0 & 1 & 0 \\ 0 & \cdots & & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix}$$

centered differences
cannot compute y_0' or y_{n+1}'
(first & last derivatives)

→ we will use centered diff. for **order 2** DE.

EX: Use centered differences to compute derivative

$$y = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} \quad \text{with} \quad h = 1/3$$

Method 1. Pointwise

$$y_1' = \frac{1}{2h}(y_2 - y_0) = \frac{3}{2}(0 - 1) = -3/2$$

$$y_2' = \frac{1}{2h}(y_3 - y_1) = 3/2(-1 - 2) = -9/2$$

$$y_3' = \frac{1}{2h}(y_4 - y_2) = 3/2(3 - 0) = 9/2$$

Method 2. Matrix

$$\begin{bmatrix} \cancel{y_0'} \\ y_1' \\ y_2' \\ y_3' \\ \cancel{y_4'} \end{bmatrix} = \frac{1}{2h} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} = 3/2 \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$$

EX: Use forward differences to compute deriv.

$$y = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} \quad \text{with} \quad h = 1/3$$

Method 1. Pointwise

$$y_0' = 1/h(y_1 - y_0) = 3(2 - 1) = 3$$

$$y_1' = 1/h(y_2 - y_1) = 3(0 - 2) = -6$$

$$y_2' = 1/h(y_3 - y_2) = 3(-1 - 0) = -3$$

$$y_3' = 1/h(y_4 - y_3) = 3(3 - (-1)) = 12$$

Method 2. Matrix

$$\begin{bmatrix} y_0' \\ y_1' \\ y_2' \\ y_3' \\ \cancel{y_4'} \end{bmatrix} = 1/h \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \\ -1 \\ 4 \end{bmatrix}$$

EX: Use backward differences for same computation.

Pointwise

$$y_1' = 1/h(y_1 - y_0) = 3(2 - 1) = 3$$

$$y_2' = 1/h(y_2 - y_1) = 3(0 - 2) = -6$$

$$y_3' = 3(-1 - 0) = -3$$

$$y_4' = 3(3 - (-1)) = 12$$

Matrix

$$\begin{bmatrix} \cancel{y_0'} \\ y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} = 3 \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$$

...